Minor theory for surfaces and divides of maximal signature

Pierre Dehornoy

Université de Grenoble / Institut Poncelet, Moscow

Embedded graphs, Saint-Petersburg
October 28th, 2014

joint with Sebatian Baader (Bern)

Posets

Definition

A *partially ordered set* is a set S together with an order $\leqslant$ that is transitive ($a \leqslant b, b \leqslant c \implies a \leqslant c$), reflexive ($a \leqslant a$), and anti-symmetric ($a \leqslant b, b \leqslant a \implies a = b$).

Posets

Definition

A *partially ordered set* is a set S together with an order $\leqslant$ that is transitive ($a \leqslant b, b \leqslant c \implies a \leqslant c$), reflexive ($a \leqslant a$), and anti-symmetric ($a \leqslant b, b \leqslant a \implies a = b$).

- $(\mathbb{N}, \leqslant)$

Posets

Definition

A *partially ordered set* is a set S together with an order $\leqslant$ that is transitive ($a \leqslant b, b \leqslant c \implies a \leqslant c$), reflexive ($a \leqslant a$), and anti-symmetric ($a \leqslant b, b \leqslant a \implies a = b$).

- $(\mathbb{N}, \leqslant)$
- $(\mathbb{N}, |)$

Posets

Definition

A *partially ordered set* is a set S together with an order $\leqslant$ that is transitive ($a \leqslant b, b \leqslant c \implies a \leqslant c$), reflexive ($a \leqslant a$), and anti-symmetric ($a \leqslant b, b \leqslant a \implies a = b$).

- $(\mathbb{N}, \leqslant)$
- $(\mathbb{N}, |)$
- $(\text{rooted embedded trees}, \leqslant^{\text{plan}})$

Posets

Definition

A *partially ordered set* is a set S together with an order $\leqslant$ that is transitive ($a \leqslant b, b \leqslant c \implies a \leqslant c$), reflexive ($a \leqslant a$), and anti-symmetric ($a \leqslant b, b \leqslant a \implies a = b$).

- $(\mathbb{N}, \leqslant)$
- $(\mathbb{N}, |)$
- $(\text{rooted embedded trees}, \leqslant^{\text{plan}})$
- $(\text{abstract graphs}, \leqslant^{\text{gr}})$

Minor-closed subsets

If $(S, \leqslant)$ is a poset, a property P is *minor-closed* if

$$u \leqslant v, P(v) \implies P(u).$$

Minor-closed subsets

If $(S, \leqslant)$ is a poset, a property P is *minor-closed* if

$$u \leqslant v, P(v) \implies P(u).$$

If there is no infinite descending chain, then the set $\{u \in S | \neg P(u)\}$ is a union of cones, defined by *minimal elements*.

Minor-closed subsets

If $(S, \leqslant)$ is a poset, a property P is *minor-closed* if

$$u \leqslant v, P(v) \implies P(u).$$

If there is no infinite descending chain, then the set $\{u \in S | \neg P(u)\}$ is a union of cones, defined by *minimal elements*.
The set of minimal elements forms an *antichain*.

Minor-closed subsets

If $(S, \leqslant)$ is a poset, a property P is *minor-closed* if

$$u \leqslant v, P(v) \implies P(u).$$

If there is no infinite descending chain, then the set $\{u \in S | \neg P(u)\}$ is a union of cones, defined by *minimal elements*.
The set of minimal elements forms an *antichain*.
Example:

Theorem (Pontryagin-Kuratowski, 1930)

A graph is planar if and only if it does not contain K_5 nor $K_{3,3}$ as a graph-minor.

Robertson-Seymour Theorem

If P is minor-closed, the set of $\neg\mathrm{P}$-minimal elements forms an *antichain*.

$(S, \leqslant)$ is a *well-quasi-ordering (wqo)* if there is no infinite descending chain and no infinite antichain.

Proposition

If $(S, \leqslant)$ is a wqo, then every minor-closed property is characterized by a finite number of prohibited minors.

Robertson-Seymour Theorem

If P is minor-closed, the set of $\neg\mathrm{P}$-minimal elements forms an *antichain*.

$(S, \leqslant)$ is a *well-quasi-ordering (wqo)* if there is no infinite descending chain and no infinite antichain.

Proposition

If $(S, \leqslant)$ is a wqo, then every minor-closed property is characterized by a finite number of prohibited minors.

Theorem (Kruskal, 1960)

$(\text{rooted trees}, \leqslant^{\text{plan}})$ *is a wqo.*

Robertson-Seymour Theorem

If P is minor-closed, the set of $\neg\mathrm{P}$-minimal elements forms an *antichain*.

$(S, \leqslant)$ is a *well-quasi-ordering (wqo)* if there is no infinite descending chain and no infinite antichain.

Proposition

If $(S, \leqslant)$ is a wqo, then every minor-closed property is characterized by a finite number of prohibited minors.

Theorem (Kruskal, 1960)

$(\text{rooted trees}, \leqslant^{\text{plan}})$ *is a wqo.*

Theorem (Robertson–Seymour, 1983-2004)

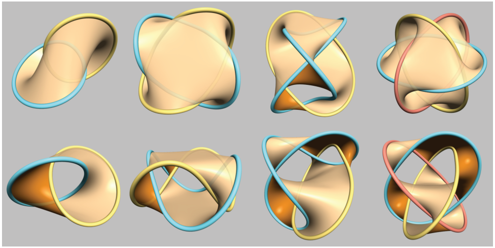
$(\text{abstract graphs}, \leqslant^{\text{gr}})$ *is a wqo.*

Embedded surfaces and surface minority

$$\mathcal{S}\mathrm{urf} = \left\{\text{embedded surfaces in } \mathbb{S}^3\right\}$$

Embedded surfaces and surface minority

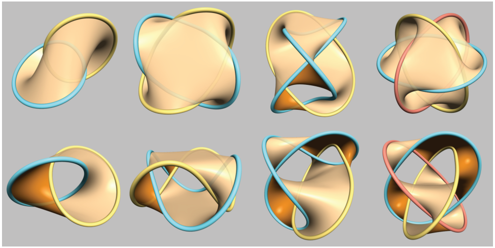
$$\mathrm{Surf} = \{\text{embedded surfaces in } \mathbb{S}^3\} \;/\; \text{isotopy}$$



(SeifertView, Jarke Van Wijk & Arjeh Cohen)

Embedded surfaces and surface minority

$\mathrm{Surf} = \{\text{embedded surfaces in } \mathbb{S}^3\} \,/\, \text{isotopy}$



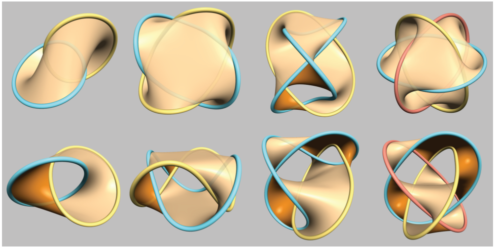
(SeifertView, Jarke Van Wijk & Arjeh Cohen)

$S' \leqslant^{\mathrm{surf}} S$ if S' incompressible subsurface of S

$(\Leftrightarrow \pi_1(S') \hookrightarrow \pi_1(S))$

Embedded surfaces and surface minority

$\mathrm{Surf} = \{\text{embedded surfaces in } \mathbb{S}^3\} \, / \, \text{isotopy}$



(SeifertView, Jarke Van Wijk & Arjeh Cohen)

$S' \leqslant^{\mathrm{surf}} S$ if S' incompressible subsurface of S

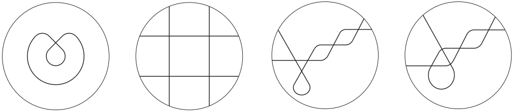
$(\Leftrightarrow \pi_1(S') \hookrightarrow \pi_1(S))$

$(\mathrm{Surf}, \leqslant^{\mathrm{surf}})$ IS NOT a wqo.

Divides

Definition (A'Campo)

A *divide* is a generic embedding of a union of segments and circles in the disc.



Divide surfaces

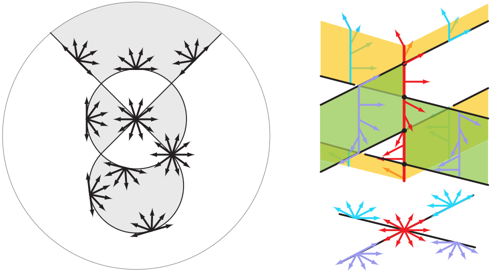
$$\mathrm{T}^1\mathrm{D}^2 \simeq \mathrm{D}^2 \times \mathbb{S}^1$$

Divide surfaces

$$T^1 D^2 \simeq D^2 \times \mathbb{S}^1$$
$$T^1 D^2 / \partial D^2 \simeq \mathbb{S}^3$$

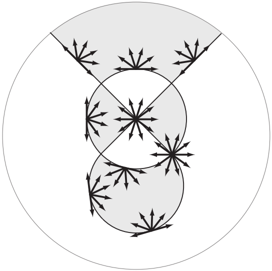
Divide surfaces

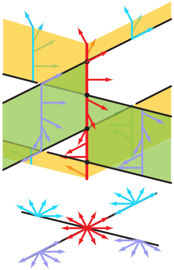
$$T^1D^2 \simeq D^2 \times \mathbb{S}^1$$
$$T^1D^2/\partial D^2 \simeq \mathbb{S}^3$$



Divide surfaces

$$T^1D^2 \simeq D^2 \times \mathbb{S}^1$$
$$T^1D^2/\partial D^2 \simeq \mathbb{S}^3$$





Theorem (A'Campo)

Divide surfaces are fiber surfaces.

Divide surfaces

$\mathcal{SD}iv = \{\text{fiber surfaces of divides}\}$.

Theorem (Baader–D, 2012)

$(\mathcal{SD}iv, \leqslant^{\text{surf}})$ *is a wqo.*

Divide surfaces

$\mathcal{SD}iv = \{\text{fiber surfaces of divides}\}$.

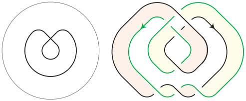
Theorem (Baader–D, 2012)

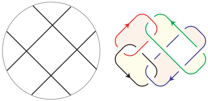
$(\mathcal{SD}iv, \leqslant^{\text{surf}})$ *is a wqo.*

Example:

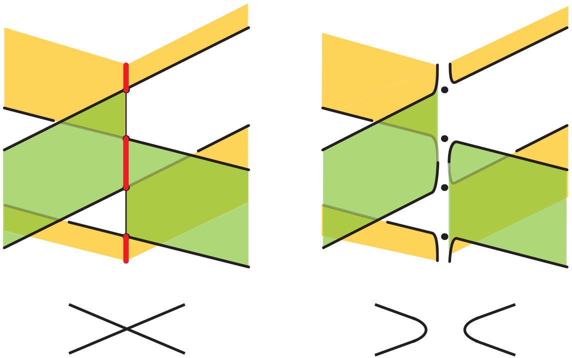
Theorem (Baader–D, 2012)

A divide surface S *satisfies* $\text{sign}(S) = b_1(S)$ *if and only if it does not contain* $S(Q)$ *nor* $S(X)$ *as a surface-minor.*

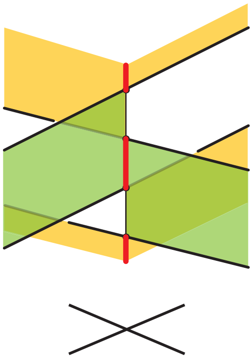




Proof: divide minors



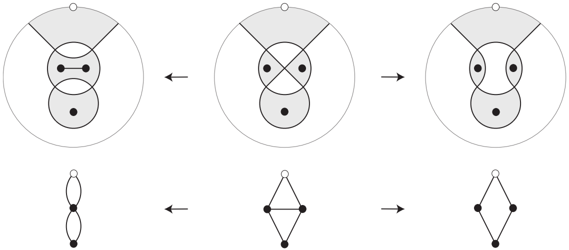
Proof: divide minors



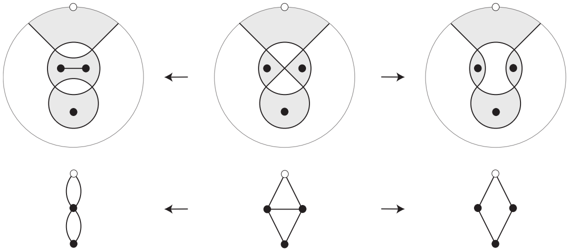
Lemma

If D' is a divide-minor of D, then $S(\mathrm{D}')$ is a surface-minor of $S(\mathrm{D})$.

Proof: planar minors



Proof: planar minors



Lemma

If G' is a planar-minor of G, then $\operatorname{div}(G')$ is a divide-minor of $\operatorname{div}(G)$.

Proof: planar minors

Proposition (Baader–D)

$(\mathrm{planar\ graphs}, \leqslant^{\mathrm{plan}})$ *is a wqo.*

Proof: planar minors

Proposition (Baader–D)

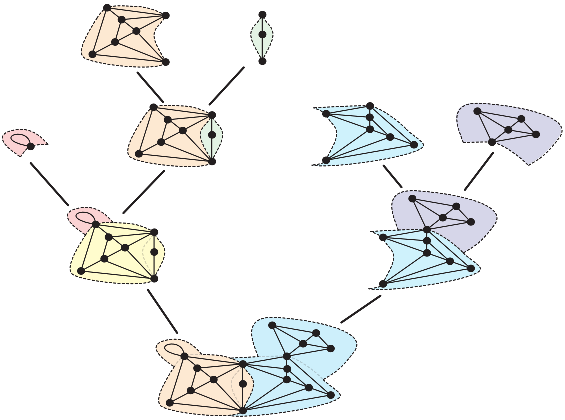
$(\mathrm{planar\ graphs}, \leqslant^{\mathrm{plan}})$ *is a wqo.*

If G' is a planar-minor of G, then $div(G')$ is a divide-minor of $div(G)$.
If $div(G')$ is a divide-minor of $div(G)$, then $S(div(G'))$ is a surface-minor of $S(div(G))$.

$\leqslant^{\mathrm{plan}}$ is a wqo

Theorem (Tutte)

A planar graph is a "tree" of 3-connected graphs.



$\leqslant^{\mathrm{plan}}$ is a wqo

Theorem (Tutte)

- *A planar graph is a “tree” of 3-connected graphs.*

$\leqslant^{\mathrm{plan}}$ is a wqo

Theorem (Tutte)

- *A planar graph is a "tree" of 3-connected graphs.*
- *3-connected graphs embed uniquely into the 3-sphere (up to reflexion).*

$\leqslant^{\mathrm{plan}}$ is a wqo

Theorem (Tutte)

- *A planar graph is a "tree" of 3-connected graphs.*
- *3-connected graphs embed uniquely into the 3-sphere (up to reflexion).*

Theorem (Kruskal, 1960)

$(\mathrm{rooted\ trees}, \leqslant^{\mathrm{plan}})$ *is a wqo.*

Theorem (Robertson–Seymour)

$(\mathrm{graphs}, \leqslant^{\mathrm{gr}})$ *is a wqo.*

Questions

Which classes of surfaces are wqo?
fiber surfaces of positive braids?
stronly-quasi positive surfaces?